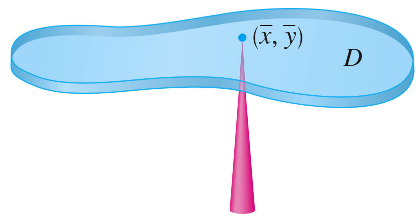
Lesson 25. Applications of Integration: Center of Mass

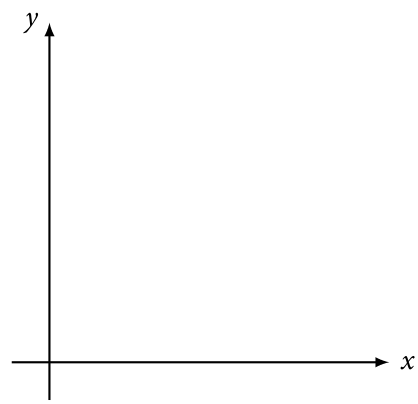
1 Definitions



- Suppose we have a **lamina** or thin plate that occupies a region D of the xy-plane
- $\rho(x, y)$ = **density** of the plate at point (x, y) (units: mass per unit area)

Example 1.

Consider a rectangular lamina with vertices $(0, 0)$, $(3, 0)$, $(0, 2)$ and $(3, 2)$ with density function $\rho(x, y) = x + y$. Where is the lamina more dense?



- The **mass** of a lamina is given by

Example 2. Find the mass of the rectangular lamina described in Example 1.

- The **moment of a lamina about the x-axis** is

- The **moment of a lamina about the y-axis** is

- The **center of mass** of a lamina is $(\overline{x}, \overline{y})$, where

  - The lamina behaves as if the entire mass is concentrated at its center of mass

Example 3. Find the center of mass of the rectangular lamina described in Example 1.

$$\begin{aligned}
M_x &= \iint_D y\rho(x, y)\, dA \\
&= \int_0^3 \int_0^2 y(x+y)\, dy\, dx \\
&= \int_0^3 \int_0^2 (xy + y^2)\, dy\, dx \\
&= \int_0^3 \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{y=0}^2 dx \\
&= \int_0^3 \left[\left(\frac{4x}{2} + \frac{8}{3}\right) - (0+0)\right] dx \\
&= \int_0^3 \left(2x + \frac{8}{3}\right) dx \\
&= \left[x^2 + \frac{8}{3}x\right]_{x=0}^3 \\
&= \left(9 + \frac{8}{3}(3)\right) - (0+0) \\
&= 17
\end{aligned}$$

2 Examples

Example 4. Find the center of mass of a lamina that is bounded by the parabolas $y = x^2$ and $x = y^2$ if the density at any point is proportional to its distance from the x-axis. Just set up the integrals, do not evaluate.

Example 5. The boundary of a lamina consists of the semicircle $y = \sqrt{4 - x^2}$ together with the x-axis. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin. Use polar coordinates. Just set up the integrals, do not evaluate.

3 If we have time...

Example 6. A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the second quadrant. Find its center of mass if the density at any point is proportional to its distance from the x-axis. Use Cartesian or polar coordinates. Just set up the integrals, do not evaluate.